QUASIPOSITIVE PRETZELS

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Abstract. A necessary and sufficient condition for an oriented pretzel surface to be quasipositive yields an estimate for the slice genus of the boundary of an arbitrary oriented pretzel surface.

§1. Introduction

A *braidzel* (see §2) is a generalized pretzel surface, with braiding data supplementing the twisting data that specifies an ordinary pretzel surface (a braidzel with trivial braiding). *Quasipositive* Seifert surfaces (see §3) can be characterized as those generated from $D^2 \subset S^3$ by plumbing positive Hopf annuli $A(O,-1)$ and passing to π_1-injective subsurfaces; they are analogues in S^3 (and, in a sense, special cases) of pieces of complex plane curve in $D^4 \subset \mathbb{C}^2$.

Theorem. *The oriented pretzel surface $P(t_1,\dots,t_k)$ is quasipositive if and only if the even integer t_i+t_j is less than 0 for $1 \le i < j \le k$.*

This result (announced in [7] for $k=3$ and t_i odd) is deduced in §4 from results on oriented braidzels. An estimate for the slice genus of the boundary of an oriented pretzel surface is given in §5. Other applications will appear elsewhere [0].

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§2. Braidzels

Let $E_k(F) := \{\{w_1,\dots,w_k\} \subset F : 0 \ne \prod_{i\ne j} w_i - w_j\}$ be the configuration space of unordered k-tuples of elements of the field F. A *k-string braid group* is the fundamental group $B_X := \pi_1(E_k(\mathbb{C});X)$ with arbitrary basepoint $X \in E_k(\mathbb{C})$; of course, every k-string braid group is isomorphic to the standard k-string braid group $B_k := B_{\{1,\dots,k\}}$ but it is very convenient to allow more general basepoints. Typically (for $k>2$) no isomorphism between distinct k-string braid groups has much claim to be called canonical, but (because $E_k(\mathbb{R})$ is contractible, as is easily seen) if $X_0, X_1 \in E_k(\mathbb{R})$ then there is a canonical homotopy class of paths in $E_k(\mathbb{C})$ from X_0 to X_1 (namely, those that stay in $E_k(\mathbb{R})$), and thus a canonical isomorphism from B_{X_0} to B_{X_1}. Consequently, any path $p : ([0,1];0,1) \to (E_k(\mathbb{C});X_0,X_1)$ represents a well-defined braid $\beta(p) \in B_k$—in particular, for any $X \in E_k(\mathbb{R})$

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FIGURE 1. $\Gamma(p)$, where $\beta(p) = \sigma_2\sigma_1\sigma_2\sigma_3\sigma_4\sigma_2\sigma_5\sigma_3\sigma_1 \in B_6$.

there are well-defined braids Δ_X, δ_X and ϱ_X in B_X which are in canonical correspondance with $\Delta_k := (\sigma_{k-1})(\sigma_{k-2}\sigma_{k-1})\cdots(\sigma_1\sigma_2\cdots\sigma_{k-1})$, $\delta_k := \sigma_{k-1}\sigma_{k-2}\cdots\sigma_1$, and $\varrho_k := \Delta_k\delta_k\Delta_k^{-1} = \sigma_1\sigma_2\cdots\sigma_{k-1}$ in B_k. Let $\gamma(p) := \{(z,\theta) : z \in p(\theta), \theta \in [0,1]\} \supset X_0 \times \{0\} \cup X_1 \times \{1\}$, $\Gamma(p) := [\min X_0, \max X_0] \times \{0\} \cup \gamma(p) \cup [\min X_1, \max X_1] \times \{1\}$ (Figure 1). For $X_0' \subset X_0$ with $k' := \operatorname{card} X_0' > 0$, the union of the components of $\gamma(p)$ which intersect $X_0' \times \{0\}$ is $\gamma(p')$ for a suitable $p' : ([0,1];0,1) \to (E_{k'}(\mathbb{C}); X_0', X_1')$, and $\Gamma(p') \subset \Gamma(p)$. Note that $\beta = \beta(p)$ determines $\beta(p') =: \beta|_{X_0'}$; in particular, if $X_0' = \{x_i, x_j\}$, $x_i \neq x_j$, then $c(x_i, x_j; \beta) \in \mathbb{Z}$ is well-defined by $\beta|_{X_0'} = \sigma_1^{c(x_i,x_j;\beta)}$.

A *braidzel* is any (unoriented, and not necessarily orientable) 2-manifold-with-boundary $S \subset \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 \subset S^3$ which has a $(0,1)$-handle decomposition

$$S = \bigcup_{s\in\{0,1\}} h_s^{(0)} \cup \bigcup_{x\in X_0} h_x^{(1)} \tag{2.1}$$

related as follows to a (piecewise-smooth) path $p : ([0,1];0,1) \to (E_k; X_0, X_1)$ with $X_0, X_1 \in E_k(\mathbb{R})$:

(2.1.0) $h_s^{(0)} \subset \mathbb{R} \times \{\theta : s + (-1)^s\theta \le 0\}$ and $h_s^{(0)} \cap (\mathbb{C} \times \{s\}) \supset [\min X_s, \max X_s] \times \{s\}$;

(2.1.1) the component of $\gamma(p)$ containing $(x,0)$ is a core arc of $h_x^{(1)}$.

The *braiding* of S is $\beta_S := \beta(p) \in B_k$; the *twisting* of S is the function $t_S : X_0 \to \mathbb{Z}$ such that

(2.1.2) $h_x^{(1)}$ makes $t_S(x)$ counterclockwise half-twists between $h_x^{(1)} \cap h_0^{(0)}$ and $h_x^{(1)} \cap h_1^{(0)}$ (its two attaching arcs).

Clearly S is orientable if and only if $t_S(x) + t_S(x') \in 2\mathbb{Z}$ for all $x, x' \in X_0$.

Though the path p to which a given braidzel S is related as in (2.1) is never unique, evidently both β_S and t_S (up to an increasing bijection of its domain) do depend on S alone, and not on p. Also evidently, every pair (β, t) is (β_S, t_S) for a braidzel well-defined up to isotopy through braidzels, and $P(\beta|_{X_0'}, t|X_0')$ can be taken to be a subbraidzel of $P(\beta, t)$ for $\varnothing \neq X_0' \subset X_0$. It is customary for a braidzel $P(o,t)$ with trivial braiding $o = o^{(k)} := 1_{B_k}$ (and, implicitly, with $X_0 = X_1$) to be called a *pretzel surface*; in one tradition, followed below, the notation for $P(o,t)$ is $P(t_1, \ldots, t_k)$, where $X_0 =: \{x_1, \ldots, x_k\}$, $x_1 < \cdots < x_k$, and $t_i := t(x_i)$.

Since $\Gamma(p)$ is a spine of $P(\beta,t)$ for all t, a diagram of $\Gamma(p)$ with the components of $\gamma(p)$ labelled according to t may be used to portray $P(\beta,t)$ schematically (Figure 2).

For $Z \in E_k(\mathbb{C})$, write $\pi : B_Z \to \operatorname{Aut}(Z)$ for the natural homomorphism; for $z \in Z$, let $\langle z \rangle : Z \to \{0,1\} \subset \mathbb{Z}$ denote the characteristic function of $\{z\}$. Let $P(\beta,t)$ be a braidzel related as in (2.1) to a path $p : ([0,1];0,1) \to (E_k; X, Y)$.

2.2. Lemma. *Each of the braidzels* (2.2.1–3) (*resp.,* (2.2.4–6)) *is isotopic to* $P(\beta,t)$ *by an isotopy supported in a regular neighborhood of* $h_0^{(0)}$ (*resp.,* $h_1^{(0)}$).

FIGURE 2. $P(3,-5,-7)$ (in full-dress; schematic).

FIGURE 3. $N_{S^3}(h_0^{(0)})$; 1-handles sliding along $\partial h_0^{(0)}$; $P(\delta_X\beta, t\circ\pi(\delta_X)+2\big\langle\max X\big\rangle)$; $P(\delta_X\beta, t\circ\pi(\varrho_X^{-1})-2\big\langle\max X\big\rangle)$.

FIGURE 4. $P(\beta,t)$; $P(\delta_X\beta, t\circ\pi(\delta_X)-2\big\langle\max X\big\rangle)$; $P(\varrho_X\beta, t\circ\pi(\varrho_X)+2\big\langle\max X\big\rangle)$; $P(\Delta_X\beta, t+1)$ (schematics).

(2.2.1) $P(\delta_X\beta, t\circ\pi(\delta_X)+2\big\langle\max X\big\rangle)$
(2.2.2) $P(\varrho_X^{-1}\beta, t\circ\pi(\varrho_X^{-1})-2\big\langle\max X\big\rangle)$
(2.2.3) $P(\Delta_X\beta, t+1)$
(2.2.4) $P(\beta\delta_Y, t+2\big\langle\pi(\beta^{-1})(\max Y)\big\rangle)$
(2.2.5) $P(\beta\varrho_Y^{-1}, t-2\big\langle\pi(\beta^{-1})(\max Y)\big\rangle)$
(2.2.6) $P(\beta\Delta_Y, t+1)$

Proof. For (2.2.1–2), the required isotopy begins with a slide of all the 1-handles counterclock-

FIGURE 5. An isotopy from $S *_\alpha A(O,-1)$ to $-(-S *_\alpha A(O,-1))$.

wise along $\partial h_0^{(0)}$; next, for (2.2.1) (resp., (2.2.2)) the 1-handle originally containing $(\min X, 0)$ is pulled up in back (resp., in front) of $h_0^{(0)}$ and acquires a positive (resp., negative) kink; finally, the kink is replaced with a half-twist counterlockwise (resp., clockwise), bringing the isotoped surface back into braidzel position. For (2.2.3), the required isotopy is a rotation of $h_0^{(0)}$ through one half-twist clockwise. (See Figures 3 and 4 for (2.2.1–3).) Similar isotopies work for (2.2.4–6). □

Remarks. (1) Of course $P(\delta_X^{-1}\beta, t \circ \pi(\delta_X^{-1}) - 2\langle \max X \rangle)$, ..., $P(\beta\Delta_Y^{-1}, t-1)$ are also isotopic to $P(\beta, t)$. (2) It is well known (and obvious) that the pretzel surfaces $P(t_1, \dots, t_k)$, $P(t_2, \dots, t_k, t_1)$, and $P(t_k, t_{k-1}, \dots, t_1)$ are mutually isotopic. This also follows immediately from (2.2) and the preceding remark, since $o^{(k)} = \delta_k o^{(k)} \delta_k^{-1} = \Delta_k o^{(k)} \Delta_k^{-1}$.

§3. Plumbing and quasipositivity

The isotopy type of $P(t_1, t_2)$ depends only on $t_1 + t_2$. Any Seifert surface isotopic to $P(t, t)$ (endowed with either orientation) is denoted $A(O, t)$, and called a *t-twisted unknotted* annulus in S^3 ("unknotted" because its core circle—equivalently, either component of its boundary—is an unknot O, and "t-twisted" because the Seifert self-linking of $[O] \in H_1(A(O,t); \mathbb{Z})$ is t). In particular, $A(O,-1)$ is called a *positive Hopf annulus* (*Hopf* because the oriented link $\partial A(O,-1)$ consists of two fibers of a Hopf fibration $S^3 \to S^2$, *positive* because the linking number of these fibers is $+1$).

There is an extensive literature on *Murasugi sum* (or *Stalling plumbing*) and its various special cases (*annular plumbing*, *arborescent plumbing*, etc.); further details are given in [8] and [10], and references cited therein. For present purposes it suffices to recall *positive Hopf plumbing* briefly.

Let $\alpha \subset S$ be a proper arc on a Seifert surface $S \subset S^3$. Let $C_\alpha \subset S$ be a regular neighborhood of α (naturally endowed with the structure of a 4-gon whose edges are alternately arcs on ∂S and proper arcs parallel to α). Let $D_\alpha \subset S^3$ be a 3-cell on the positive side of S with $D_\alpha \cap S = C_\alpha$. Let $A(O,-1) \subset D_\alpha$ be a positive Hopf annulus with $A(O,-1) \cap \partial D_\alpha = C_\alpha$ such that $\partial A(O,-1) \subset \partial C_\alpha$ consists of the two proper arcs parallel to α. Say that $S \cup A(O,-1) =: S *_\alpha A(O,-1)$ is constructed by *plumbing* $A(O,-1)$ *to* S *along* α, and that S is constructed from $S *_\alpha A(O,-1)$ by *deplumbing* $A(O,-1)$. Up to isotopy, $S *_\alpha A(O,-1)$ depends only on S (up to isotopy) and α (up to isotopy on S); moreover (Figure 5), $S *_\alpha A(O,-1)$ is isotopic to $-(-S *_\alpha A(O,-1))$, where $-S$ denotes the surface S with orientation reversed.

A subsurface T of a surface S is π_1*-injective* if, whenever a simple closed curve $C \subset T$ bounds a disk on S, then C already bounds a disk on T. The notation $T \Subset S$ (or $S \Supset T$) will indicate that T is a π_1-injective subsurface of S. For example, if S is constructed from S' by deplumbing $A(O,-1)$, then $S \Subset S'$.

The notion of *quasipositivity*, introduced in [3] and [4], has been elaborated and explored in a

number of later papers. In particular, [5] and [8] taken together justify the following definition: a Seifert surface $S \subset S^3$ is *quasipositive* if and only if, up to isotopy, S is constructed from D^2 by a sequence of moves, each of which is either plumbing $A(O, -1)$ or passing to a π_1-injective subsurface.

Remarks. (1) In fact, [5] shows that, if S is quasipositive, then there exists such a sequence of moves in which all the positive Hopf plumbings come first (and can even be done simultaneously, to appropriate arcs on D^2, thus creating a positive Hopf-plumbed *basket* in the sense of [8], [10]), followed by a single passage to a π_1-injective subsurface (called a *full* subsurface in [5]); the present formulation is more convenient. (2) Also, [8] shows that arbitrary Murasugi sums with quasipositive plumbands (not just positive Hopf plumbings $S *_\alpha A(O, -1)$ with S quasipositive) are quasipositive; that generality is not needed here.

§4. Characterization of quasipositive pretzel Seifert surfaces

Let $P(\beta, t)$ be an orientable braidzel (endowed with either orientation) related as in (2.1) to a path $p : ([0, 1]; 0, 1) \to (E_k; X, Y)$.

4.1. Lemma. *If $k = 2$, so $X = \{x_1, x_2\}$ and $\beta = \sigma_1^m = \Delta_2^m$, then $P(\beta, t)$ is quasipositive if and only if $t(x_1) + t(x_2) < 2m$.*

Proof. If $m = 0$, then $P(\beta, t)$ is a $(t(x_1) + t(x_2))/2$-twisted unknotted annulus, and the conclusion follows from [6]. If $|m| > 0$, then $P(\beta, t)$ is isotopic (by $|m| - 1$ applications of (2.2.3)) to $P(t(x_1) - m, t(x_2) - m)$. □

4.2. Corollary. *If $P(\beta, t)$ is quasipositive, then $t(x_i)+t(x_j) < 2c(x_i, x_j; \beta)$ for all $x_i, x_j \in X$, $x_i \neq x_j$. In particular, if $P(t_1, \ldots, t_k)$ is quasipositive, then $t_i + t_j < 0$ for $1 \leq i < j \leq k$.* □

Call $t : X \to \mathbb{Z}$ *nearly negative* if $t \leq 0$ and $\mathrm{card}\{x : t(x) = 0\} \leq 1$.

4.3. Proposition. *If an oriented pretzel surface has nearly negative twisting, then it is quasipositive.*

Proof. If $P(t_1, \ldots, t_k)$ is an oriented pretzel surface with nearly negative twisting, then its boundary is a *positive link*, that is, it has a diagram (namely, the full-dress diagram for $P(t_1, \ldots, t_k)$ in the style of Figure 2) in which every crossing is positive. By [2] or [9], the application of Seifert's original algorithm [11] to this diagram produces a quasipositive Seifert surface S, which by inspection is $P(t_1, \ldots, t_k)$. □

Remarks. (1) Of course the boundary of an oriented pretzel surface with $t \leq 0$ is always a positive link, but if $\mathrm{card}\{x : t(x) = 0\} > 1$ then Seifert's algorithm produces a surface which is disconnected, and so not $P(t_1, \ldots, t_k)$. (2) For variety and to illustrate other techniques, here is an alternative proof using the Twist Insertion Lemma [3] (Figure 6 indicates a "coordinate-free" statement and proof of this lemma independent of the machinery of braided surfaces applied in [3]). If either all t_i are -1, or one t_i is 0 and the rest are -2, an easy induction on k establishes that $P(t_1, \ldots, t_k)$ is isotopic to a quasipositive Hopf plumbed basket

$$(\ldots (D^2 *_{\alpha_1} A(O, -1)) *_{\alpha_2} A(O, -1) \ldots) *_{\alpha_{k-1}} A(O, -1),$$

for appropriate proper arcs $\alpha_1, \ldots, \alpha_{k-1} \subset D^2$ (see Figure 7). In general, if $t_i \leq 0$ for $i = 1, \ldots, k$, and $t_i = 0$ for at most one i, then $P(t_1, \ldots, t_k)$ can (depending on the parity of the t_i)

FIGURE 6. The regular neighborhood of an arbitrary proper arc on a quasipositive Seifert surface S; S', produced by plumbing two positive Hopf annuli to S, is also quasipositive; so is $S'' \Subset S'$; but S'' is isotopic to S with a clockwise full twist inserted.

FIGURE 7. Plumbing arc arrangements for basket presentations of $P(-1, \ldots, -1)$ and $P(0, -2, \ldots, -2)$.

FIGURE 8. A braidzel $P(\beta_0\sigma_i, t)$; a Seifert surface S isotopic to $P(\beta_0\sigma_i, t)$; a Seifert surface $S' * A(O, -1) \Supset S$; a braidzel $P(\beta_0, t)$ isotopic to S'.

be produced from either $P(-1, \ldots, 1)$ or (at least) one of $P(0, -2, \ldots, -2)$, $P(0, -2, \ldots, -2)$, $\ldots$, $P(-2, -2, \ldots, 0)$, by introducing extra clockwise full twists into some of the 1-handles, so the Twist Insertion Lemma shows that $P(t_1, \ldots, t_k)$ is quasipositive. $\square$

Call a braid $\beta \in B_k$ *non-negative* if it can be written as a word in the standard generators $\sigma_1, \ldots, \sigma_{k-1} \in B_k$ with no negative exponents. (A non-negative braid in which each σ_i appears non-trivially is conventionally called *positive*.)

4.4. Proposition. *If β is non-negative and t is nearly negative, then $P(\beta, t)$ is quasipositive.*

Proof. The case of trivial braiding $\beta = o^{(k)}$ is 4.3. The proof for general non-negative $\beta \in B_k$ is established by induction on the length of a word for β in $\sigma_1, \ldots, \sigma_{k-1} \in B_k$ without negative exponents. The proof of the inductive step (illustrated in Figure 8) consists in showing, by a sequence of handle slides supported in a regular neighborhood of $h_1^{(0)}$, that $P(\beta_0\sigma_i, t)$ (resp., $P(\beta_0, t)$) is isotopic to a Seifert surface S (resp., S') with $S \Subset S' * A(O, -1)$. $\square$

4.5. Theorem. *An oriented pretzel surface $P(t_1,\dots,t_k)$ is quasipositive if and only if $t_i+t_j<0$ for $1\le i<j\le k$.*

Proof. "Only if" was proved in 4.2, and "if" with the extra hypothesis that t be nearly negative in 4.3. If $P(t_1,\dots,t_k)$ is an oriented pretzel surface with $t_i+t_j<0$ for $1\le i<j\le k$, and t is not nearly negative, then there is exactly one i with $t_i>0$, and $t_j<0$ for $j\ne i$; without loss of generality (by the second remark after 2.2), $t_1>0$. By (2.2.3) and (2.2.2), $P(t_1,t_2,\dots,t_k)$ is isotopic to $P(\Delta_k\varrho_k^{-1},t')$, where $\Delta_k\varrho_k^{-1}=(\sigma_{k-1})(\sigma_{k-2}\sigma_{k-1})\dots(\sigma_2\sigma_3\dots\sigma_{k-1})$ is non-negative, $t'(1)=t_1-1\ge 0$, and $t'(j)=t_{k-1-j}+1<0$ for $2\le j\le k$; a sequence of t_1 such moves gives an isotopy from $P(t_1,t_2,\dots,t_k)$ to $P(\beta,t'')$, where $\beta=(\Delta_k\varrho_k^{-1})^{t_1}$ is non-negative and t'' is nearly negative. By 4.4, $P(t_1,t_2,\dots,t_k)$ is quasipositive. $\square$

§5. Slice estimates

Recall that, if $L\subset S^3=\partial D^4$ is an oriented link, then $\chi_s(L)$ denotes the greatest Euler characteristic $\chi(F)$ such that $F\subset D^4$ is a smooth surface with $L=\partial F$ such that no component of F has empty boundary. If K is a knot, then its *slice* (or *Murasugi*) *genus* $g_s(K)$ (the least genus $g(F)$ for such an F with $K=\partial F$) equals $(1-\chi_s(K))/2$.

The following results are proved in [7] and [8].

5.1. Proposition. *If Q is a quasipositive Seifert surface, then $\chi_s(\partial Q)=\chi(Q)$. If S is any Seifert surface, and $Q\subset S$ is quasipositive, then $\chi_s(\partial S)\le 2\chi(Q)-\chi(S)$.* $\square$

Let $P(\beta,t)$ be an oriented braidzel related to $p:([0,1];0,1)\to(E_k;X,Y)$.

5.2. Proposition. *If β is non-negative and t is nearly negative, then $\chi_s(\partial P(\beta,t))=2-k$. More generally, for any $X'\subset X$ such that $\beta|X'$ is non-negative and $t|X'$ is nearly negative, $\chi_s(\partial P(\beta,t))\le 2-2\operatorname{card}X'+k$.* $\square$

Proof. Immediate from 4.4 and 5.1. $\square$

Let $P(t_1,\dots,t_k)$ be an oriented pretzel surface.

5.3. Proposition. *If $t_i+t_j<0$ for $1\le i<j\le k$, then $\chi_s(\partial P(t_1,\dots,t_k))=2-k$. More generally, for any $(t_1,\dots,t_k)$, $\chi_s(\partial P(t_1,\dots,t_k))\le 2+\operatorname{card}\{i:t_i\ge 0\}-\operatorname{card}\{j:t_j<0\}-\varepsilon$, where $\varepsilon=1$ if $\min\{t_i:t_i\ge 0\}+\max\{t_j:t_j<0\}<0$, $\varepsilon=0$ otherwise.*

Proof. Immediate from 4.5 and 5.1. $\square$

Remarks. (1) Let $\mathrm{Mir}:S^3\to S^3$ be an orientation-reversing diffeomorphism, so for any oriented $M\subset S^3$, $\mathrm{Mir}\,M$ is the mirror image of M. Since $\chi_s(L)=\chi_s(\mathrm{Mir}\,L)$ for any L, the application of 5.3 to $P(-t_k,\dots,-t_1)=\mathrm{Mir}\,P(t_1,\dots,t_k)$ gives another upper bound on $\chi_s(\partial P(t_1,\dots,t_k))$. (2) The oriented pretzel surface $P(t_1,\dots,t_k)$ is bounded by a knot iff k and $t_1,\dots,t_k$ are all odd. In this case $g_s(\partial P(t_1,\dots,t_k))$ is readily bounded below using 5.3 (and the first remark). In particular, it is easy to recover the non-sliceness results of Yu and Kuga for certain algebraically slice pretzel knots [12] and connected sums thereof [1], obtained by methods of "classical" gauge theory (in contrast to the methods used to prove the local Thom Conjecture, an appeal to the truth of which underlies [7] and thus the present paper).

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